Grade 7/8 Math Circles
November 6/7/8/9
Geometric Sequences

Squares in Rectangles in Squares in...

We'll begin with an activity and look out for some patterns.

Exercise 1

(i) Begin by drawing a square with side lengths of 1 unit.

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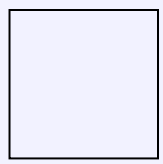
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What is the area of the square?

(ii) Now, divide your square into half so you have two equal rectangles. Shade in one of the halves.

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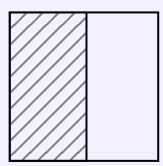
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What is the area of each half? What is the area of the shaded region?

(iii) Again, divide the unshaded rectangle into two halves. Shade in one of these new halves.

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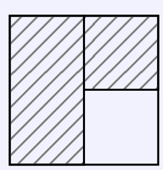
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What is the area of each half? What is the total area of all the shaded regions?

Exercise 1 Continued

(iv) Repeat step (iii) two more times, each time dividing the unshaded region into two, finding the area of each new half, and the total area of the shaded regions.

(v) What patterns do you notice as you're finding the areas? Without actually drawing the rectangles, can you predict what the area of each new half will be after we divide the square 6 times? 10 times?

What about the shaded regions? Can you predict what the total area of the shaded regions will be after we divide the square 6 times? 10 times?

Recall

Before we examine the solutions, let's quickly review exponents. Just like multiplication represents repeated addition, that is

$$2 \times 4 = 2 + 2 + 2 + 2$$

we have that exponents represent repeated multiplication:

$$2^4 = 2 \times 2 \times 2 \times 2$$

Exercise 2

Calculate the following exponents: 3^4, 2^6, and $\left(\frac{1}{2}\right)^3$. Note that $\left(\frac{1}{2}\right)^3 = \frac{1}{2^3}$

Solution

(i) The area of the square is

$$Area = length \times width = 1 \times 1 = 1$$

(ii) Since we divided the square into halves, each rectangle has width of $\frac{1}{2}$, so

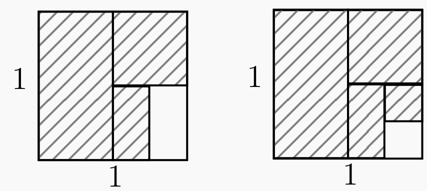
$$Area\ of\ half\ = Area\ of\ shaded = \frac{1}{2} \times 1 = \frac{1}{2}$$

(iii) Now, the length is half of what it was before, so we have

$$\textit{Area of half} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\textit{Area of shaded} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

(iv) The next two divisions should look similar to this



The left square has areas

$$\textit{Area of half} = \frac{1}{8} \qquad \textit{Area of shaded} = \frac{7}{8}$$

and the right square has areas

$$\textit{Area of half} = \frac{1}{16} \qquad \textit{Area of shaded} = \frac{15}{16}$$

(v) One might notice that at each step, we divide the previous *Area of half* by 2 (or multiply it by $\frac{1}{2}$) and that *Area of shaded* $= 1 -$ *Area of half*.

Repeatedly multiplying by $\frac{1}{2}$ can be represented using exponents; if we multiply by $\frac{1}{2}$ three times, we have $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$, which is exactly what we got after 3 divisions. We can use this to predict the areas.

After dividing the square 6 times, we should have

$$\textit{Area of half} = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

and after 10 times,

$$\textit{Area of half} = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

This is a tiny number! We would not want to try and divide the square 10 times to find this number. Using these numbers, we can find the areas of the shaded regions after 6 times

$$\textit{Area of shaded} = 1 - \textit{Area of half} \; = 1 - \frac{1}{64} = \frac{63}{64}$$

and after 10 times

$$\textit{Area of shaded} = 1 - \textit{Area of half} \; = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Note that in the formulas above, we always know that the

$$\textit{Area of half} \; = \left(\frac{1}{2}\right)^n$$

where n is the number of times we divide the circle. Then, since *Area of shaded* $= 1 -$ *Area of half*, we can write

$$\textit{Area of shaded} = 1 - \left(\frac{1}{2}\right)^n$$

Now we can find the area of the shaded regions without needing to calculate the area of each half first!

Sequences

A **sequence** is an ordered list of numbers. For example, the sequence of integers from 1 to 10 is written as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Each number in the sequence is called a **term**. Terms in our sequence are separated by commas.

A sequence is either *finite* or *infinite*. A *finite* sequence will eventually end, like the one above, but an infinite sequence goes on forever. For example, the sequence of odd numbers

$$\{1, 3, 5, 7, 9, 11, ...\}$$

is an infinite sequence. We use an ellipsis (...) to show that we are omitting information, since it's impossible to write down every term in an infinite sequence.

To refer to a specific term of the sequence, we use t_n, with the subscript n representing the n^{th} term in the sequence. For example, t_1 represents the first term in a sequence.

Exercise 3

Identify t_2, t_5, and t_8 in the infinite sequence below:

$$\{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, ...\}$$

Geometric Sequences

You may have noticed a pattern in the sequence from Exercise 2, to get to the next term, we just multiply the previous term by 2. This value is called the **common ratio**.

A **geometric sequence** is a sequence that has a common ratio, meaning that if you take any term in the sequence and multiply it by the common ratio, you will get the next term in the sequence. The common ratio can be any number. For example, it could be a positive number, a negative number, a fraction, etc.

Exercise 4

Identify the geometric sequences from the list below. For any sequence that is geometric, also state the common ratio.

(i) $\left\{\frac{1}{2}, 1, 2, 4\right\}$

(ii) $\{5, 10, 15, 20, 25, ...\}$

(iii) $\{1, 5, 1, 5, 1, 5, 1, 5, ...\}$

(iv) $\{3, 9, 27, 81\}$

(v) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, ...\right\}$

Does the last sequence in Exercise 3 look familiar? It is exactly the values that we got for the areas of the rectangles in Example 1 each time that we divided the rectangle into halves. So dividing areas forms geometric sequences!

Sums and Series

One thing that we can do with sequences in general (not just geometric sequences!) is add up some or all of the terms in the sequence. This is called a **series**. The result that you get after adding up the terms is called the *sum*. For example, for the sequence $\{1, 2, 3, 4, 5\}$. The series of this sequence is

$$1 + 2 + 3 + 4 + 5$$

and the sum is 15 since

$$1 + 2 + 3 + 4 + 5 = 15$$

Note that a series can be either finite or infinite, just like a sequence.

Exercise 5

For each sequence in Exercise 3, write out its series and find the sum. Since we cannot add together an infinite series by ourselves, if the sequence is infinite, find the sum of the first 5 terms.

If our sequence is a geometric sequence, then we call its series a **geometric series**.

Example 1

Recall the sequence for the area of the rectangles in Exercise 1:

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\}$$

We saw that this is a geometric sequence with common ratio $\frac{1}{2}$. To calculate the sum of the first 5 terms, we can do it by hand, as in Exercise 1,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$$

or we can use the formula that we found, *Area of shaded* $= 1 - \left(\frac{1}{2}\right)^n$, with $n = 5$ as follows:

$$1 - \left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$$

and notice that we get the same sum!

We really like working with finite geometric series because there is a formula that we can use that will find the sum for us. The formula is:

$$a \times (1 - r^n) \div (1 - r)$$

where a is the first term in the sequence, r is the common ratio, and n is the number of terms in our sum.

Example 2

Consider the geometric sequence

$$\{1, 2, 4, 8, 16, 32\}$$

We have that the first term is 1, so $a = 1$. The common ratio is 2, so $r = 2$. The number of terms is 6, so $n = 6$. Using our formula for a geometric sum, we get

$$\begin{aligned} a \times (1 - r^n) \div (1 - r) &= 1 \times \left(1 - 2^6\right) \div (1 - 2) \\ &= 1 \times (1 - 64) \div (1 - 2) \\ &= 1 \times (-63) \div (-1) \\ &= 63 \end{aligned}$$

We can verify that our formula is correct by adding

$$1 + 2 + 4 + 8 + 16 + 32 = 63$$

Exercise 6

For the geometric sequence

$$\{2, 6, 18, 54\}$$

find a, r, and n. Write out the it's series and find its sum using the formula for the sum of a geometric series.

Additionally, for each geometric sequence in Exercise 4, verify that our formula gives us the same sum as the sums you found in Exercise 5.

Notice that this formula looks very different than the one used in our Example 1. That is, for the geometric sequence

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots\right\}$$

we had that the formula for the sum of its geometric series was

$$1 - \left(\frac{1}{2}\right)^n$$

However, it turns out that, with the help of some algebra, one can show that

$$1 - \left(\frac{1}{2}\right)^n = a \times (1 - r^n) \div (1 - r)$$

where $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

Exercise 7

By trying different values of n, verify that the two formulas above give the same sums.

Infinite Series

Up until now, whenever our sequence has been infinite, we never find the sum of its series. Instead, we've only found the sum of the first n terms. But what if we want to know what happens when we add together an infinite number of terms?

In the formula for the sum of a geometric series, we have the term r^n. This term is very important for whether or not we can find the sum of an infinite geometric series.

Stop and Think

What happens to 2^n as n gets bigger and bigger?

Hopefully, you noticed that each time we increase n, the value of 2^n gets much larger. In fact, 2^n can get so large that we are no longer able handle the numbers. Further, this happens *any time* that $r \geq 1$. Because of this, we are unable to calculate the sum of a geometric series when $r \geq 1$.

Stop and Think

What happens to $\left(\frac{1}{2}\right)^n$ as n gets bigger and bigger? Think about what happened to the area of the rectangles in Exercise 1 each time that we divided them into halves.

If we were to keep dividing the rectangles in Exercise 1 into halves, they would eventually become so small that we cannot even see them, let alone draw them. In fact, if we were to divide the rectangles 20 times, then each half would have an area of

$$\left(\frac{1}{2}\right)^{20} = 0.00000095367431641$$

This is so small that we can basically pretend that it is 0. In fact, if we want n to be arbitrarily large, then it really is 0, not just pretend! So for arbitrarily large n, $\left(\frac{1}{2}\right)^n = 0$. Now, lets finally try to find the series of our geometric sequence

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\}$$

Using our formula from before with $a = \frac{1}{2}, r = \frac{1}{2}$, and n being arbitrarily large, we have

$$\begin{aligned} a \times (1 - r^n) \div (1 - r) &= \frac{1}{2} \times \left(1 - \left(\frac{1}{2}\right)^n\right) \div \left(1 - \frac{1}{2}\right) \\ &= \frac{1}{2} \times (1 - 0) \div \left(1 - \frac{1}{2}\right) \\ &= \frac{1}{2} \times 1 \div \frac{1}{2} \\ &= 1 \end{aligned}$$

We've successfully added together an infinite number of terms and finally found the sum of an infinite geometric series! In fact, this works *any time* that $r < 1$. That is, when we have $r < 1$, we can find the sum of the infinite geometric series.

What is 0.99999999...?

When working with fractions and decimals, you have likely came across fractions that have infinite decimal representations. For example,

$$\frac{1}{3} = 0.\dot{3} = 0.333333...$$

A particular infinite decimal that gets a lot of attention is $0.\dot{9} = 0.999999....$

Stop and Think

If we were to represent $0.\dot{9}$ as a whole number or fraction, what should this number be? Why do you think this?

Exercise 8

What are the decimal representations of the following fractions?

1. $\frac{9}{10}$
2. $\frac{9}{100}$
3. $\frac{9}{1000}$
4. $\frac{9}{10000}$

We should have gotten $\frac{9}{10} = 0.9$, $\frac{9}{100} = 0.09$, $\frac{9}{1000} = 0.009$, and $\frac{9}{10000} = 0.0009$. Notice what happens when we add together these fractions:

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} = 0.9 + 0.09 + 0.009 + 0.0009 = 0.9999$$

Notice that this is a geometric sequence with common ratio $\frac{1}{10}$! Let's return our attention back to $0.\dot{9}$. Notice the following:

$$\begin{aligned} 0.\bar{9} = 0.999999... &= 0.9 + 0.09 + 0.009 + 0.0009 + 0.00009 + 0.000009 + \cdots \\ &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \frac{9}{100000} + \frac{9}{1000000} + \cdots \end{aligned}$$

So we've turned our infinite decimal into a geometric series with common ratio $\frac{1}{10}$. Since the common ratio is less than 1, we know exactly how to deal with this! We'll use our formula for geometric series with $a = \frac{9}{10}, r = \frac{1}{10}$, and let n be arbitrarily large to get

$$\begin{aligned} a \times (1 - r^n) \div (1 - r) &= \frac{9}{10} \times \left(1 - \left(\frac{1}{10}\right)^n\right) \div \left(1 - \frac{1}{10}\right) \\ &= \frac{9}{10} \times (1 - 0) \div \left(1 - \frac{1}{10}\right) \\ &= \frac{9}{10} \times 1 \div \frac{9}{10} \\ &= 1 \end{aligned}$$

So we have that $0.\dot{9} = 0.999999... = 1$, very interesting!